Longitudinal and lateral dispersion in an isotropic porous medium

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Coefficients of longitudinal and lateral dispersion were measured for steady uniform laminar flow through an isotropic porous medium. A unique experimental method for measuring lateral dispersion is described. It is found that the ratio of the coefficient of longitudinal dispersion D_1 to the coefficient of lateral dispersion D_2 is given by $D_1/D_2 = \lambda \Re^n$, where λ and n are dimensionless coefficients dependent upon the pore-system geometry, and $\Re$ is the Reynolds number based on the seepage velocity, the average grain diameter, and the kinematic viscosity.

1. Introduction

Although it is known that mass transport perpendicular to a uni-directional flow through a porous medium (lateral dispersion) is less than mass transport in the direction of the flow (longitudinal dispersion), little quantitative information is available in the literature. It is the purpose of this paper to report an analytical and experimental investigation of dispersion in an isotropic porous medium which gives the magnitude of the longitudinal and lateral dispersion coefficients for a wide range of Reynolds numbers.

For steady uniform flow parallel to the x-axis in an isotropic homogeneous porous medium, the equation for the conservation of the mass of the dispersing substance can be written as (Scheidegger 1961; Harleman & Rumer 1962)

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = D_1\frac{\partial^2 s}{\partial x^2} + D_2\frac{\partial^2 s}{\partial y^2},\tag{1}$$

where s = concentration of dispersing mass; u = seepage velocity = (volume rate of flow)/(cross sectional area of voids) [cm/sec]; x, y = co-ordinate directions [cm]; D_1 = coefficient of longitudinal dispersion [cm^2/sec]; D_2 = coefficient of lateral dispersion [cm^2/sec]; t = time [sec].

It has been well established by experiments that the mechanical process of longitudinal dispersion is a function of both the geometry of the pore system and the absolute magnitude of the seepage velocity in the Darcy flow range. It should be mentioned that for very small seepage velocities the transport of dispersing mass by molecular diffusion becomes important relative to the mechanical mixing of the dispersion process. For longitudinal dispersion, molecular diffusion has been experimentally determined (Raimondi, Gardner & Petrick 1959) to be important at Reynolds numbers less than 10^{-3}.

Scheidegger (1961) postulates that the coefficients of dispersion are related to the pore-system geometry and the seepage velocity by the expressions

$$D_1 = \alpha_1 |u| \tag{2}$$

and
$$D_2 = \alpha_2 |u|, \tag{3}$$

where α_1 and α_2 are coefficients having the dimensions of length and dependent upon the pore-system geometry. The absolute value of the seepage velocity is used since negative dispersion coefficients have no meaning. Some investigators (Raimondi *et al.* 1959; Mehlhorn 1962) have found that α_1 is proportional to the average grain size of a porous medium, i.e.

$$\alpha_1 = \beta d, \tag{4}$$

where β is a constant of proportionality presumably dependent upon the particle shape and size distribution and d is the average grain size (the 50 % size taken from a standard gravimetric sieve analysis). Substituting equation (4) into equation (2) and dividing by the kinematic viscosity, ν, gives

$$D_1/\nu = \beta \Re, \tag{5}$$

where $\Re = ud/\nu$.

In a recent paper, Harleman, Mehlhorn & Rumer (1963) it was found that a more general relationship given by

$$D_1/\nu = \psi_1 \Re^{n_1} \tag{6}$$

is a better representation of experimental data for longitudinal dispersion. It was shown that ψ_1 is independent of particle size for nearly uniform media and that ψ_1 is dependent on particle shape and possibly size distribution. The exponent n_1 equals 1·20 and is independent of particle size and shape for nearly uniform media. Preliminary observations indicate that n_1 is a function of size distribution and that it may approach unity for highly non-uniform media.

A similar postulate using equation (3) leads to a relation for the lateral dispersion coefficient,
$$D_2/\nu = \psi_2 \Re^{n_2}. \tag{7}$$

Hence, the ratio of the longitudinal and lateral dispersion coefficients should be of the form
$$D_1/D_2 = \lambda \Re^{n}, \tag{8}$$

where $\lambda = \psi_1/\psi_2$ and $n = n_1 - n_2$. In accordance with the above discussion λ and n should be functions only of the media characteristics.

De Jong (1958), in an analytical study, predicted the ratio of D_1/D_2 to be a function of the media characteristics and the distance travelled by the tracer in a steady uniform flow. Relatively little experimental data on lateral dispersion of the type described by equation (1) is found in the literature. Blackwell (1959) reported some radial transverse dispersion experiments in packed columns. His results showed that mass transport by molecular diffusion became important at higher seepage rates in transverse dispersion as compared to longitudinal dispersion. Blackwell found the ratio D_1/D_r to be dependent upon the magnitude of the seepage velocity; however, the ratio was independent of the distance travelled by the tracer. Simpson (1962) reported experiments measuring the

lateral dispersion of a line source in a steady uniform flow through a porous medium; however, dispersion coefficients of the form given by equation (1) were not determined.

It is the purpose of this study to investigate the validity of equations (6), (7) and (8) in a porous medium composed of nearly uniform plastic spheres.

2. Theoretical considerations

(a) *Longitudinal dispersion*

Consider a steady uniform flow through a column packed with a porous medium. At time $t = 0$ a constant concentration, s_0, of a tracer substance is introduced into the flow such that at $x = 0$ the tracer concentration is always s_0 (see figure 1).
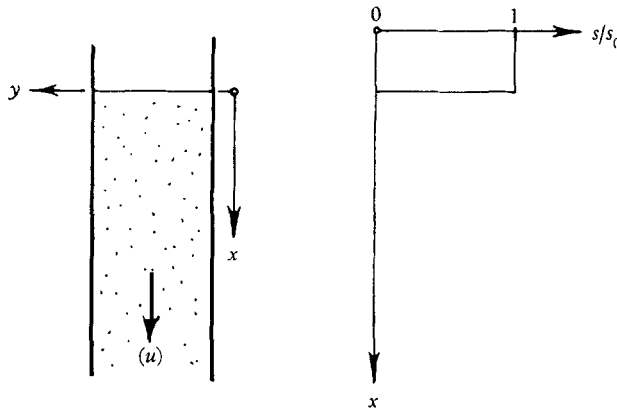

FIGURE 1. Longitudinal dispersion at $t = 0$.

For this case there is no gradient of the tracer concentration in the y-direction and equation (1) becomes

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = D_1\frac{\partial^2 s}{\partial x^2}. \tag{9}$$

The boundary and initial conditions are

$$\left.\begin{aligned}
s(0, t) &= s_0 \quad (t \geqslant 0);\\
s(x, 0) &= 0 \quad (x > 0);\\
s(\infty, t) &= 0 \quad (t \geqslant 0).
\end{aligned}\right\} \tag{10}$$

The solution to equation (9) with the indicated boundary and initial conditions has been given as (Rifai, Kaufman & Todd 1956; Ogata & Banks 1961)

$$\frac{s}{s_0} = \tfrac{1}{2}\operatorname{erfc}\left(\frac{x - ut}{2(D_1 t)^{\frac{1}{2}}}\right) + \tfrac{1}{2}\exp\left(\frac{ux}{D_1}\right)\operatorname{erfc}\left(\frac{x + ut}{2(D_1 t)^{\frac{1}{2}}}\right). \tag{11}$$

It has been shown by Ogata & Banks that the second term of equation (11) can be neglected with an error of less than 5 % when D_1/ux is less than 0·0075. This restriction is satisfied in the experiments reported in this investigation.

(b) Lateral dispersion

Consider a steady uniform flow through a porous medium contained between impermeable boundaries (see figure 2). The seepage velocity u is constant throughout the medium. At $x = 0$ the concentration is held constant and equal to s_0 for $y < 0$ and equal to zero for $y > 0$. If the impermeable boundaries are far enough apart so that the concentration along the bottom boundary remains s_0 for all x
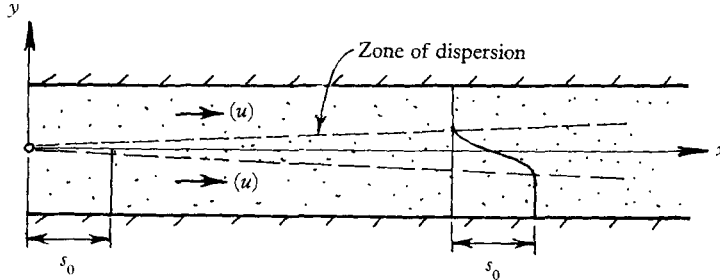

FIGURE 2. Sketch of lateral dispersion process.

and along the top boundary remains zero for all x, the following boundary conditions may be applied to the steady-state distribution of the tracer concentration.

$$
\left.
\begin{aligned}
s(0, y) &= s_0 \quad (-\infty < y \leqslant 0); \\
s(0, y) &= 0 \quad (0 < y < +\infty); \\
\partial s/\partial y &= 0 \quad (y = \pm\infty \quad \text{for all } x).
\end{aligned}
\right\}
\tag{12}
$$

For this case equation (1) reduces to

$$
u\,\partial s/\partial x = D_2(\partial^2 s/\partial y^2).
\tag{13}
$$

The assumption that $\partial^2 s/\partial x^2 \ll \partial^2 s/\partial y^2$ was experimentally verified. The solution to equation (13) with the indicated boundary conditions is obtained by use of the Fourier transform of s. The solution is

$$
\frac{s}{s_0} = \tfrac{1}{2}\operatorname{erfc}\left(\frac{y}{2(D_2 x/u)^{\frac{1}{2}}}\right).
\tag{14}
$$

Equation (14) can be written as

$$
\operatorname{erf}\left(\frac{y}{2(D_2 x/u)^{\frac{1}{2}}}\right) = 1 - 2\frac{s}{s_0}.
\tag{15}
$$

The quantity $(1 - 2s/s_0)$ can be computed from experimental measurements for each value of x and y. Thus the quantity $y/2(D_2 x/u)^{\frac{1}{2}}$ can be determined from tables of the error function. For each value of x, y and u the quantity $y/2(x/u)^{\frac{1}{2}}$ can be computed. If a best-fit straight line is drawn through the experimentally determined values of $y/2(D_2 x/u)^{\frac{1}{2}}$ and $y/2(x/y)^{\frac{1}{2}}$ on rectilinear co-ordinate paper the slope of this line is equal to $1/D_2^{\frac{1}{2}}$.

3. Experimental equipment and procedure

(a) Porous medium

The porous medium used was composed of Dow Chemical 'Pelaspan 8' plastic spheres. The spheres had a uniformity coefficient (ratio of 60 % size to 10 % size) of 1·14, an effective size (10 % size) of 0·86 mm, and a mean size of 0·96 mm. The porosity of the plastic sphere packings was 0·36. The average permeability was 0·8 cm/sec.
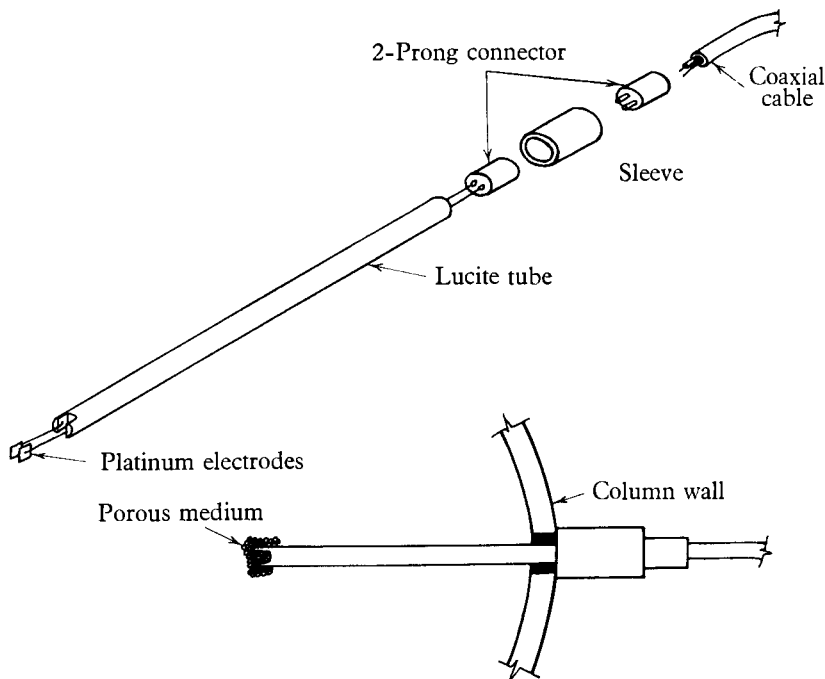

FIGURE 3. Conductivity probe, type I.

(b) Longitudinal dispersion

The experimental equipment and procedure for the longitudinal dispersion tests has been described in detail in another paper (Rumer 1962). The experimental procedure consisted of measuring, by conductivity methods, the concentration of sodium chloride at a fixed point in the column as a function of time. The signal from the conductivity cell was recorded on an oscillograph incorporating an independent timer. Dilute aqueous solution of less than 0·1 % by weight of sodium chloride were used for the tracer concentration, s_0. The conductivity probe (referred to as type I) is shown in figure 3.

(c) Lateral dispersion

The lateral dispersion experiments were performed in a Lucite box 302 cm long with a cross-section 15·2 cm high and 7·62 cm wide (see figure 4). The medium was packed into the box, covered with a sponge rubber mat, and the top was

secured by bolting through flanges which ran along the sides of the box. The compressed sponge rubber prevented short-circuiting of the flow along the top of the box. The boundary condition of equation (12) was provided by means of a
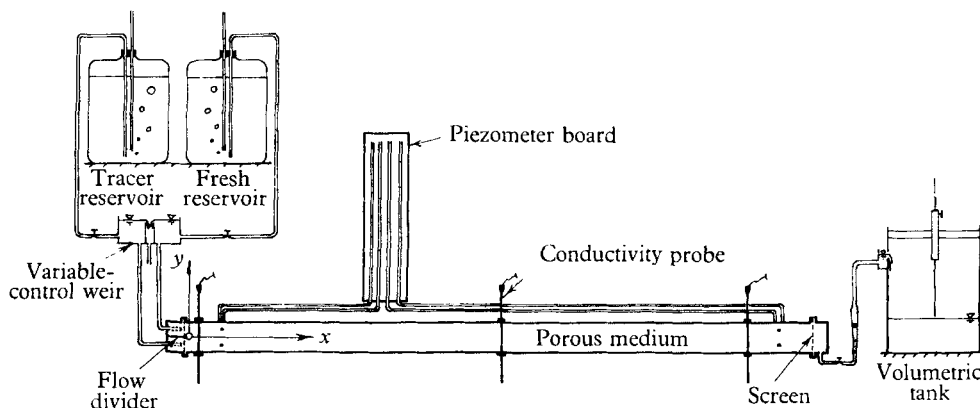

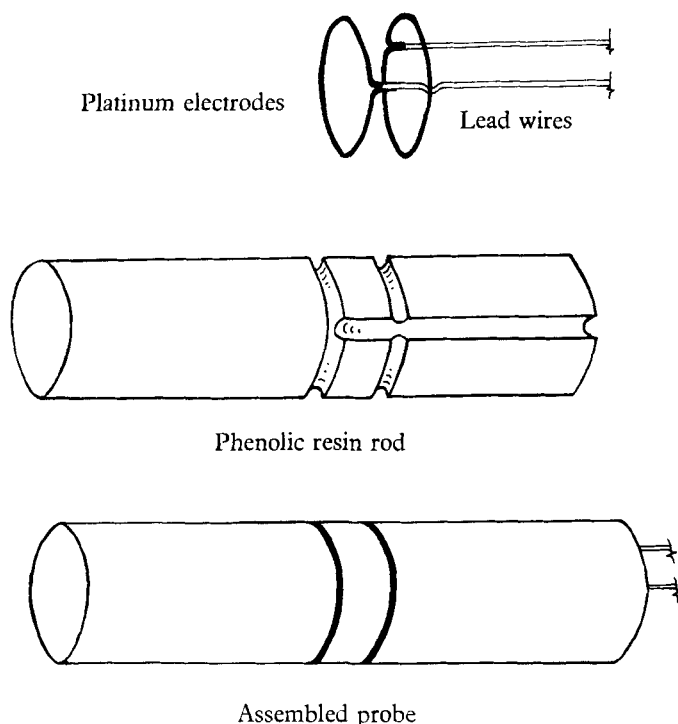FIGURE 4. Schematic diagram of lateral dispersion apparatus.

Assembled probe

FIGURE 5. Conductivity probe, type II.

thin brass plate which kept the fresh-water flow and the tracer-solution flow separated for x less than zero.

Since, for the lateral dispersion experiments, the conductivity-cell position in the y-direction had to be adjustable, the probe used in the longitudinal dispersion experiments was not suitable. A probe was needed that would not be

sensitive to rotation and that would cause negligible disturbance to the flow and the medium when moved. The probe shown in figure 5 (referred to as type II) was developed. All lead wires were cemented and shielded with epoxy-cement. The wire electrodes (wire diameter $= 0.5$ mm) were located 2.5 mm apart. Probes were located in the box at $x = 5$ cm, 151.5 cm, and 273.5 cm.

The procedure for the lateral dispersion experiments was as follows. The fresh water and tracer solution were permitted to flow until a steady-state zone

Longitudinal dispersion		
Run no.	D_1 [cm²/sec]	u [cm/sec]
L–5	0·00826	0·130
L–6	0·0018	0·0363
L–7	0·000224	0·00608
L–8	0·00361	0·0679
L–9	0·00165	0·0325
L–10	0·00663	0·101

Lateral dispersion		
Run no.	D_2 [cm²/sec]	u [cm/sec]
1–L	0·000208	0·0575
2–L	0·000256	0·0802
3–L	0·000384	0·103
4–L	0·000137	0·0295
5–L	0·0000925	0·0116
7–L	0·000256	0·0603
8–L	0·000256	0·0635
9–L	0·000538	0·175
10–L	0·000635	0·203
11–L	0·000656	0·284
12–L	0·000296	0·0826

TABLE 1. Tabulation of experimental results.

of dispersion had been established. An approximate indication of this time was the length of time for a particle moving at the seepage velocity to traverse the length of the box. When steady-state conditions prevailed, the vertical profiles of tracer concentration were measured at the three stations. It was necessary to wait a short interval of time after each probe movement in order to flush the vicinity around the electrodes of any fluid dragged from its previous position. The isochlor $(s/s_0 = 0.5)$ never departed more than 1 cm from the x-axis and generally was within a range of ± 0.5 cm. This deviation was not considered excessive. Sugar was added to the fresh water so that the density of the tracer solution and the fresh water were the same.

4. Presentation and discussion of results

The experimental data are tabulated in table 1. No noticeable variation of the lateral dispersion coefficients with distance was found. For all runs the standard deviation of the concentration profile for each value of x correlated linearly

with $x^{\frac{1}{2}}$ as predicted by equation (14). Typical results for a lateral dispersion test can be seen in figure 6.

The correlation of D_1/ν and D_2/ν with Reynolds number is shown in figure 7. Equation (6) becomes

$$D_1/\nu = 0 \cdot 66 \Re^{1 \cdot 2}$$

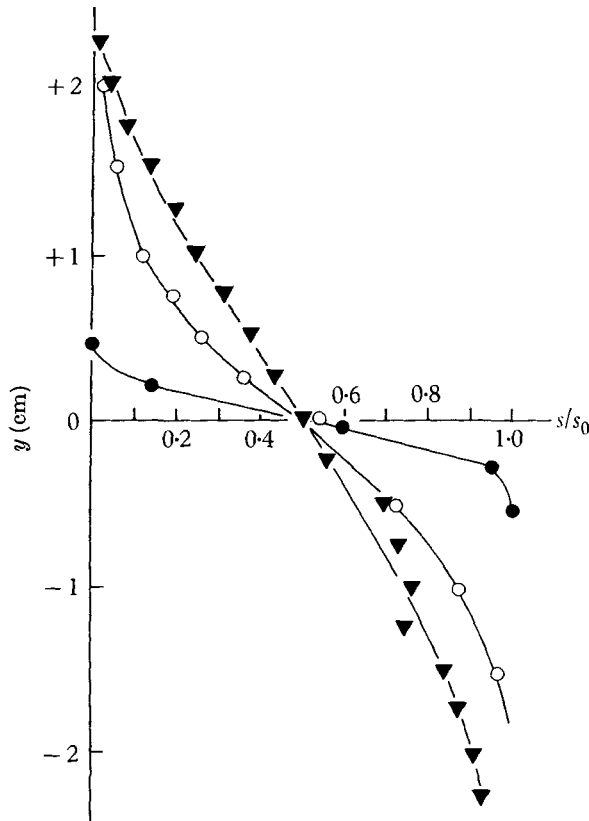and equation (7) becomes

$$D_2/\nu = 0 \cdot 036 \Re^{0 \cdot 7}.$$

FIGURE 6. Concentration profiles for a lateral dispersion test. $u = 0 \cdot 0635$ cm/sec. ●, $x = 5$ cm; ○, $x = 151 \cdot 5$ cm; ▼, $x = 273 \cdot 5$ cm.

At Reynolds numbers less than 10^{-2} the curves appear to converge and they would become horizontal at D/ν equal to the Schmidt number, where D would then be the effective molecular diffusivity in porous media. Equation (8) becomes

$$D_1/D_2 = 18 \cdot 3 \Re^{0 \cdot 5}$$

for the porous medium investigated. For a Reynolds number equal to $0 \cdot 1$, $D_1/D_2 = 5 \cdot 8$ and at a Reynolds number equal to one, $D_1/D_2 = 18 \cdot 3$. It can be seen that the ratio increases with increasing seepage velocity. The extension of this equation for Reynolds numbers greater than ten is not justified. Bischoff & Levenspiel (1962) indicate that as the flow moves out of the Darcy range into the so-called non-linear range, the exponent n_1 decreases and the exponent n_2 increases. This transition has not been well established experimentally although

it would appear that the increased inertial forces and the onset of turbulence are the causes for the changes in the exponents.
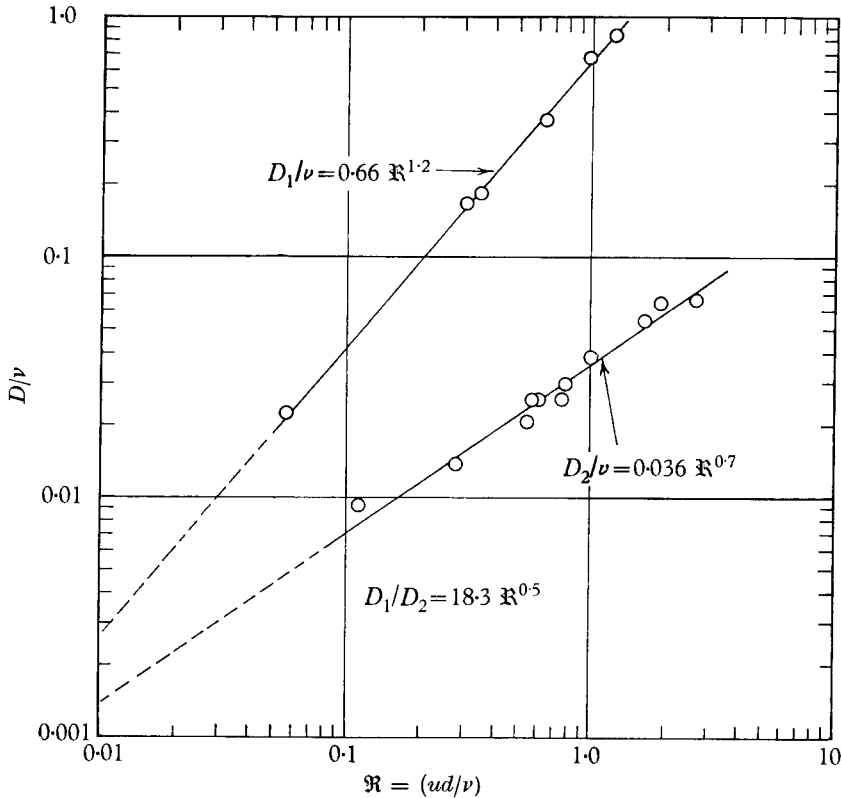

FIGURE 7. Correlation of dispersion coefficients with Reynolds number.

5.   Conclusions

The functional relationships for the longitudinal and lateral dispersion coefficients in a porous medium set forth in equations (6) and (7) are valid in the laminar flow range above a Reynolds number of the order of 10^{-3}. At lower Reynolds numbers mass flux due to molecular diffusion becomes important relative to the mechanical mixing of the dispersion process. The equations also appear to delineate the effect of particle size and shape for uniform media.

For the isotropic homogeneous medium investigated, the ratio of the longitudinal coefficient of dispersion to the lateral coefficient of dispersion depends upon the absolute magnitude of the seepage velocity and is independent of distance. This ratio is represented in equation form as

$$D_1/D_2 = \lambda \Re^n = 18\cdot3\Re^{0\cdot5},$$

where λ is a dimensionless coefficient presumably dependent upon particle shape for uniform media and also on particle size distribution for non-uniform media. The exponent n has been found to be equal to $0\cdot5$ in this investigation.

Much remains to be investigated in the phenomenon of dispersion. Lateral dispersion should be investigated over a wide range of seepage velocities,

particularly for Reynolds numbers greater than ten. Non-uniform media composed of angular particles and anisotropic media remain to be investigated, as well as the influence of a density difference on the lateral mixing.

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